Northern Beaches Secondary College

Manly Selective Campus

2012 HSC Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators and templates may be used.
- All necessary working should be shown in every question.
- Multiple choice questions are to be completed on the special answer page.

Total marks – 70

- Attempt Questions 1-14.

Marks

Multiple choice questions - answer on the special answer sheet provided.

1. A class consists of 10 girls and 12 boys. How many ways are there of selecting a committee of 3 girls and 2 boys?

(A) 7920

(B) 26334

(C) 95040

(D) 3160080

2. Which of the following is an expression for $\int \frac{x}{\sqrt{x^2-2}}dx$.

Use the substitution $u = x^2 - 2$.

(A) $\sqrt{x^2-2}+c$.

(B) $\left(\sqrt{x^2-2}\right)^3+c$.

(C) $2\sqrt{x^2-2}+c$.

(D) $2\left(\sqrt{x^2-2}\right)^3+c$.

3. Seven people are seated around a round table. How many arrangements are possible if two people refuse to sit beside each other?

(A) 24

(B) 120

(C) 240

(D) 480

4. The expression $\tan\left(\frac{\pi}{4}+x\right) =$ can be also expressed as

(A) $\dfrac{\cos x+\sin x}{\cos x-\sin x}$

(B) $\dfrac{\cos x-\sin x}{\cos x+\sin x}$

(C) $\dfrac{\sec^2 x}{1-\tan^2 x}$

(D) $\dfrac{\sin x+\cos x}{\sin x-\cos x}$

5. Determine the acute angle between the lines $2x + y = 4$ and $x - 3y = 2$ to the nearest degree.

(A) 82^0

(B) 45^0

(C) 18^0

(D) 72^0

6. What is the exact value of the definite integral $\displaystyle\int_{\frac{2}{\sqrt{3}}}^{2\sqrt{3}} \frac{dx}{x^2+4}$?

(A) $\frac{\pi}{12}$

(B) $\frac{\pi}{6}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

7. Which of the following is true for the equation of motion $y = 3\sin\frac{2}{3}t$?

(A) period = $\frac{2}{3}$.

(B) period = $\frac{4}{3}$.

(C) period = 3.

(D) the period is none of the above values.

8. $\lim\limits_{x\to 0} 3x \operatorname{cosec}\frac{x}{2} =$

(A) $\frac{2}{3}$.

(B) $\frac{1}{6}$.

(C) $\frac{3}{2}$.

(D) 6.

9. The length of each side of a cube is increasing at a rate of 1 mm/min. At what rate is the volume increasing when the side edge is 10 cm?

(A) 6 cm^3/min

(B) 12 cm^3/min.

(C) 30 cm^3/min

(D) none of the above values.

Marks

10. Which of the following expressions is correct?

(A) $\tan^{-1}x = \cos^{-1}\dfrac{1}{\sqrt{1-x^2}}$

(B) $\tan^{-1}x = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$

(C) $\tan^{-1}x = \cos^{-1}\dfrac{x}{\sqrt{1-x^2}}$

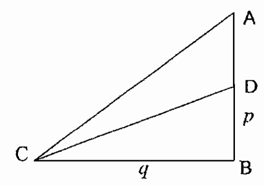
(D) $\tan^{-1}x = \cos^{-1}\dfrac{x}{\sqrt{1+x^2}}$

Marks

Free response questions – answer each question in a separate Booklet

Question 11 (Start a new Booklet) **15**

(a) In ΔABC, $A\hat{B}C = 90°$ and CD bisects $A\hat{C}B$. $BD = p$ and $BC = q$.



Find an expression for sin $A\hat{C}B$ in terms of p and q. *(2)*

(b) What is the monic polynomial Q(x) whose roots are *(2)*

$$2+\sqrt{3}\ ,\ 2-\sqrt{3}\ \text{ and }\ 2$$

Express your answer in expanded form.

(c) Using Newton's method with $x = 3$, find a better approximation for the value of x in the equation $\tan x = \log_e x$. *(2)*

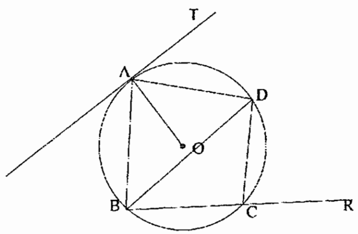
Marks

Question 11 (continued)

(d) Points A, B, C and D lie on a circle with centre O.
The line TA is a tangent to the circle at A and BC is produced to R.

The interval OA bisects $B\hat{A}D$ and BC = CD.

Let the size of $D\hat{B}C = \alpha$.



(i) Show $O\hat{A}D = \alpha$. *(2)*

(ii) Prove that $A\hat{B}C$ is a right angle. *(2)*

(e) (i) By considering the sum of an arithmetic series, show that *(1)*

$$(1 + 2 + 3 + \ldots + n)^2 = \frac{1}{4}n^2(n+1)^2.$$

(ii) Hence use the technique of mathematical induction to prove that *(4)*

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2 \quad n \geq 1$$

Marks

Question 12 (Start a new Booklet) **15**

(a) $T(2t, t^2)$ is point on the parabola $x^2 = 4y$ with focus F. P is the point which divides FT internally in the ratio 1:2 . Show that the locus of P is a parabola. *(4)*

(b) Determine $\int \sin^2\left(\frac{x}{2}\right) dx$ *(2)*

(c) A cup of hot coffee at temperature $T^{o}C$ loses heat when placed in a cooler environment. It cools according to the law

$$\frac{dT}{dt} = k(T - T_0)$$

where t is time elapsed in minutes and T_o is the temperature of the environment in degrees Celsius.

(i) Show that the equation $T = T_0 + Ae^{kt}$ is a solution for the differential equation. *(1)*

(ii) A cup of coffee at 100°C is placed in an environment at -20°C for 3 minutes during which time it cools to 70°C. Find the exact value of k. *(1)*

(ii) The same cup of coffee, at 70°C, is then placed in an environment at 20°C. Assuming k remains the same, find the temperature of the coffee after a further 15 minutes. *(2)*

(d) The acceleration of a particle moving in a straight line is given by $\ddot{x} = -2e^{-x}$ where x metres is the displacement from the origin. Initially the particle is at the origin with a velocity of 2 ms^{-1}.

Prove $v = 2e^{-\frac{x}{2}}$. *(2)*

(e) Determine the coefficient of x^5 in the expansion of $\left(2x^2 + \frac{1}{x}\right)^7$. *(3)*

Marks

Question 13 (Start a new Booklet) ***15***

(a) The velocity of a particle moving along the x axis is given by

$$v^2 = -7 + 8x - x^2$$

(i) Find an expression for the acceleration of the particle. *(2)*

(ii) Explain why the motion of the particle is simple harmonic. *(1)*

(iii) Identify the amplitude, period and centre of motion. *(2)*

(iii) Find the maximum speed. *(1)*

(b) The function $f(x)$ is defined as $f(x) = \text{cosec } x \quad 0 \le x \le \frac{\pi}{2}$.

(i) State the range of f(x). *(1)*

(ii) Find $f^{-1}(x)$. *(2)*

(iii) Hence find $\frac{d}{dx}\left(f^{-1}(x)\right)$ *(2)*

(c) Draw a fully labelled sketch of $y = 4\cos^{-1}(2 - 3x)$. *(2)*

(d) Solve the following inequality for : $\frac{2}{x} \ge x - 1$. *(2)*

Marks

Question 14 (Start a new Booklet) ***15***

(a) A particle is moving in a straight line. At time t seconds it has displacement x metres to the right of a fixed point O on the line and velocity v ms^{-1} given by $v = \sin x \cos x$.
The particle starts at $\frac{\pi}{4}$ metres to the right of O.

(i) Show that $\frac{d}{dx}\ln(\tan x) = \frac{1}{\sin x \cos x}$ *(1)*

(ii) Hence show that the displacement of the particle is given by $x = \tan^{-1}\left(e^t\right)$. *(2)*

(b) A rescue plane is travelling at an altitude of 120 metres above sea level and a constant speed of 216 km/h towards a stranded sailor. A canister containing a life raft is dropped from the plane to the sailor.

(i) How long will it take for the canister to hit the water? (Take $g = 10\text{ms}^{-2}$) *(4)*

A current is causing the sailor to drift at a speed of 3.6 kmh^{-1} in the same direction as the plane is travelling. The canister is dropped when the horizontal distance from the plane to the sailor is D metres.

(ii) What values can D take if canister lands at most 50 metres away from the sailor? *(2)*

(c) Find the greatest coefficient in the expansion $(2 + 5x)^{11}$. *(2)*

(d) By considering the expansion of $\left(2 + \frac{x}{2}\right)^n$ prove that *(4)*

$$\left(\frac{5}{2^2}\right)^n - 1 = \frac{1}{2^2}\binom{n}{1} + \frac{1}{2^4}\binom{n}{2} + \ldots + \frac{1}{2^{2n}}\binom{n}{n}$$

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

Part A:

Question	*Answer*	*Working*
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

Question 11

(a)	$\tan D\hat{C}B = \frac{p}{q}$ and $\sin D\hat{C}B = \frac{p}{\sqrt{p^2+q^2}}$ But CD bisects $A\hat{C}B$ $\therefore \quad A\hat{C}B = 2 \times D\hat{C}B = 2 \times \alpha$ (say) $\therefore \quad \sin 2\alpha = 2 \times \frac{p}{\sqrt{p^2+q^2}} \times \frac{q}{\sqrt{p^2+q^2}}$ $= \frac{2pq}{p^2+q^2}$	*2 marks – correct expression.* *1 mark – recognition of need for a double angle.*
(b)	$\sum \alpha = 6$ $\sum \alpha\beta = (2+\sqrt{3})(2-\sqrt{3}) + 2(2+\sqrt{3}) + 2(2-\sqrt{3}) = 9$ $\alpha\beta\gamma = (2+\sqrt{3}) \times 2(2-\sqrt{3}) = 1 \times 2 \times 1 = 2$ monic polynomial is $x^3 - 6x^2 + 9x - 2 = 0$	*2 marks – correct polynomial* *1 mark – correct approach using roots and at least one component value correct.*
(c)	let $y = \tan x - \log_e x$ $\frac{dy}{dx} = \sec^2 x - \frac{1}{x}$ At x = 3 $y = \tan 3 - \ln 3$ $\frac{dy}{dx} = \sec^2 3 - \frac{1}{3}$ REMEMBER radians $y_2 = 3 - \frac{\tan 3 - \ln 3}{\sec^2 3 - \frac{1}{3}}$ $= 4.81$	*2 marks – correct approximation.* *1 marks – correct derivative and substitutions.*
(d)(i)	$BC = CD$ $\therefore \Delta B\hat{D}C$ is isosceles and $D\hat{B}C = B\hat{D}C = \alpha$ $R\hat{C}D = 2\alpha$ (exterior angle of triangle BCD equals the sum of interior opposite angles). $D\hat{C}R = B\hat{A}D = 2\alpha$ (external angle of a cyclic quadrilateral equals the interior opposite angle). But OA bisects $B\hat{A}D$ $\therefore O\hat{A}D = \alpha$	*2 marks – correct proof with justification.* *1 mark – correct proof with lack of sufficient justification* *OR appropriate identification of <DCR.*

(d) (ii)	As $O\hat{A}D = \alpha \quad T\hat{A}D = 90° - \alpha$ $\therefore A\hat{B}D = 90° - \alpha$ (angle between a tangent AT and a cord AD = angle in alternate segment). But $D\hat{B}C = \alpha$ $\therefore A\hat{B}C = 90°$	*2 marks – correct proof* *1 mark – correct use of tangent theorem.*
(e) (i)	arithmetic series with $t_1 = 1$ and $d = 1$ $ts\ S_n = \frac{n}{2}(2 + (n-1)) = \frac{n}{2}(n+1)$ $\therefore \quad (1 + 2 + \ldots)n)^2 = \frac{n^2}{4}(n+1)^2$	1 mark – correct proof.
(ii)	Try n = 1 $LHS = 1^3 = 1$ $RHS = 1^2 = 1 \quad \therefore LHS = RHS$ Assume true for n = k $\therefore (1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3\ldots + n)^2$ Try $\mathbf{n} = \mathbf{k} + 1$ SO RTP $1^3 + 2^3 + 3^3 \ldots + \mathbf{n}^3 + (\mathbf{n} + 1)^3 = (1 + 2 + \quad + \mathbf{k} + \mathbf{k} + 1)^2$ $= \frac{1}{4}(\mathbf{k} + 1)^2 (\mathbf{k} + 2)^2$ $LHS = (1 + 2 + 3\ldots + n)^2 + (n+1)^3$ $= \frac{1}{4}n^2(n+1)^2 + (n+1)^3$ $= \frac{1}{4}(n+1)^2(n^2 + 4n + 4) = \frac{1}{4}(n+1)^2(n+2)^2$ $= RHS$ Hence proved by technique of mathematical induction n >=1	*4 marks – correct demonstration fully set out.* *3 marks – correct substitution for assumption* *2 marks – correct RTP statement with additional statement included appropriately* *1 mark – correct statement for n = k.*

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2012 HSC Mathematics Extension 1 – assessment 3 - solutions

Question 12

(a)	Coordinates of P $P=\left(\frac{2\times 0+1\times 2t}{1+2},\ \frac{2\times 1+1+t^2}{3}\right)=\left(\frac{2t}{3},\ \frac{2+t^2}{3}\right)$ Locus of P $x=\frac{2t}{3} \qquad \frac{3x}{2}=t$ $y=\frac{1}{3}(2+t^2)$ $=\frac{1}{3}\left(2+\frac{9x^2}{4}\right)$ $12y=9x^2+8$ $12y-8=9x^2$ $x^2=\frac{4}{3}\left(y-\frac{2}{3}\right)$	
(b)	$\int \sin^2\left(\frac{x}{2}\right)dx$ $=\int \frac{1}{2}-\frac{1}{2}\cos x\, dx$ $=\frac{x}{2}-\frac{1}{2}\sin x + C$	
(c) - i	$T=T_0+Ae^{kt}$ $\therefore T-T_0=Ae^{kt}$ $\frac{dT}{dt}=tAe^{kt}$ $=t(T-T_0)$	

c- (ii)	$t=0 \qquad T_0=-20 \qquad T=100$ $T=T_0+Ae^{kt}$ $100=-20+Ae^0$ $A=120$ $T=-20+120e^{3k}$ $70=-20+120e^{3k}$ $\ln\left(\frac{3}{4}\right)=3\mathbf{k}$ $\mathbf{k}=\frac{1}{3}\ln\left(\frac{3}{4}\right)$	
c-(iii)	$t=0 \qquad T_0=20 \qquad T=70$ $T=T_0+Ae^{kt}$ $100=70+Ae^0$ $A=30$ $T=20+30e^{15k}$ $=20+120e^{15\times\ln\left(\frac{3}{4}\right)}$ $=21.6^0\ C$	

(d)	$\ddot{x} = -2e^{-x}$ $\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -2e^{-x}$ $\frac{1}{2}v^2 = 2e^{-x} + \frac{C}{2}$ $t = 0\ x = 0\ v = 2$ $4 = 4 + C$ $C = 0$ $v^2 = 4e^{-x}$ $v = \pm\sqrt{4e^{-x}}$ $= \pm 2e^{-\frac{x}{2}}$ $v = 2e^{-\frac{x}{2}}$ Velocity is initially positive and acceleration approaches 0 with increasing distance therefore velocity remains positive.	
(e)	$\left(2x^2 + \frac{1}{x}\right)^7$ $\therefore$ *General Term is* ${}^7C_k\left(2x^2\right)^{7-k}.\left(\frac{1}{x}\right)^k$ ${}^7C_k\left(2^{7-k}\right)\left(x^2\right)^{7-k}.(x)^{-k} = Ax^5$ ${}^7C_k\left(2^{7-k}\right)x^{143-k} = Ax^5$ $\therefore\quad 14 - 3k = 5$ $k = 3$ $\therefore\quad A = {}^7C_2 \times 2^4 = 336$	

Question 13

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Question 14

Question 15

Question 16
